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Massive Spin-2 fields of Geometric Origin in Curved Spacetimes

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Abstract

We study the consistency of a model which includes torsion as well as the metric as dynamical fields and has massive spin-2 particle in its spectrum. It is known that this model is tachyon- and ghost-free in Minkowski background. We show that this property remains valid and no other pathologies emerge in de Sitter and anti-de Sitter backgrounds, with some of our results extending to arbirary Einstein space backgrounds. This suggests that the model is consistent, at least at the classical level, unlike, e.g., the Fierz–Pauli theory.

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1 Introduction and summary

The aim of this note is to examine the possibility of obtaining a geometrical field theory, consistent at least at the classical level, for an interacting massive spin-2 particle. The problems with massive spin-2 particles are well illustrated by the Fierz–Pauli theory [1] of massive graviton. In Minkowski background, the latter theory is ghost-free and describes the propagation of 5 degrees of freedom, the right number for massive spin-2 particle. One of the problems arising already in Minkowski background is the van Dam–Veltman–Zakharov discontinuity [2, 3] and strong coupling related to it [4]. Even worse, once the background is not Minkowski, a new, Boulware–Deser mode starts to propagate [5, 6, 7], and this mode is necessarily a ghost. Yet another problem with the Fierz–Pauli theory is that even otherwise healthy modes exhibit superluminal propagation [8].

Some of these problems may be cured by considering theories whose "natural" backgrounds are not Minkowski [9, 10, 11] or theories with broken Lorentz-invariance [12, 13, 14, 15] (for a review see, e.g., Ref. [16]). On the other hand, there is another class of classical theories which are known to admit consistent propagation of both massless gravitons and massive spin-2 modes in Minkowski background [17, 18, 19, 20]. These theories treat the vierbein and the connection as independent variables and include, in addition to the scalar curvature of the connection, quadratic terms in the full curvature tensor and its contractions. They also include bilinear terms in the torsion which eventually act like mass terms for propagating modes emerging from the torsion. It is a theory from this class that we focus on in this note; we will specify the Lagrangian of the theory below in this Section. Our purpose is to study whether this theory exhibits the Boulware–Deser phenomenon and/or superluminal propagation in curved backgrounds.

We note that there is a subclass of theories studied in Refs. [17, 18, 19, 20] in which, besides massless graviton, only spin-0 modes propagate in Minkowski background. It has been argued [21, 22] that the models from this subclass are fully consistent. This is in agreement with the expectation that a theory whose spectrum about Minkowski background does not include massive spin-2 states may not be problematic in curved backgrounds. Unlike in Refs. [21, 22], we study here the theory that does contain propagating massive spin-2 modes in Minkowski background, and thus, off hand, has less chance to be fully consistent.

In the Fierz–Pauli theory, the Boulware–Deser mode and superluminal propagation show up already in backgrounds of the highest possible symmetry admitted in that theory [16, 8]. Thus, as the first step, it is natural to study what are the most symmetric backgrounds

admitted by the theory we are interested in, and then analyze the propagation of perturbations about these backgrounds. We find that the theory admits Einstein spaces with zero torsion as solutions to the field equations. These include the maximally symmetric, de Sitter and anti-de Sitter spaces. Our main result is that the number of propagating modes in these maximally symmetric backgrounds is exactly the same as in Minkowski space, and that these modes obey the standard massive equations of the Klein–Gordon type. Since all propagating modes are not ghosts in Minkowski background, they are not ghosts in the curved backgrounds we study. The latter property follows from continuity argument: as the parameters of our model change continuously, the background smoothly interpolates between Minkowski and (anti-)de Sitter spaces of different curvature, while the kinetic terms in the field equations do not become singular or vanish; hence the kinetic part of the action for propagating modes does not change sign. Thus, there is neither Boulware–Deser phenomenon nor superluminal propagation, at least in maximally symmetric backgrounds. This suggests that the theory we study in this note, and possibly other theories from the class described in Refs. [17, 18, 19, 20], may be consistent theories of massive spin-2 particles. It is worth pointing out that these theories have been considered in an astrophysical context with constraints on the parameters of the Lagrangian from observational data [23].

To describe the class of models of Refs. [17, 18, 19, 20], we denote the vierbein as usual by e^i_μ and the connection, which can be regarded as an $O(1,3)$ gauge field, by $A_{ij\mu} = -A_{ji\mu}$, where $\mu = 0,1,2,3$ is the space-time index and $i,j = 0,1,2,3$ are the tangent space indices. The curvature is then defined as in any Yang-Mills theory, namely[2]

$$F_{ijmn} = e^\mu_m e^\nu_n (\partial_\mu A_{ij\nu} - \partial_\nu A_{ij\mu} + A_{ik\mu} A^k{}_{j\nu} - A_{ik\nu} A^k{}_{j\mu})$$

Throughout this paper we mostly use the tangent space basis. In this basis the indices are raised and lowered by the Minkowski metric η_{ij} (signature is mostly positive).

From the vierbein e^i_μ and its inverse e^μ_i one constructs an object denoted by C_{ijk} and defined by

$$C_{ijk} = e^\mu_j e^\nu_k (\partial_\mu e_{i\nu} - \partial_\nu e_{i\mu}) \tag{1}$$

This object together with the connection $A_{ijk} = e^\mu_k A_{ij\mu}$ enables one to introduce the torsion tensor T_{ijk},

$$T_{ijk} = A_{ijk} - A_{ikj} - C_{ijk}$$

The actions studied in this paper will be a subclass of the most general actions which are quadratic in T_{ijk} as well as in F_{ijkl} and its all possible contractions with η_{ij} and ε_{ijkl}. These

[2] In this paper we closely follow the notations of Refs. [17, 18, 19].

contractions are

$$F_{jl} = \eta^{ik} F_{ijkl}, \qquad F = \eta^{jk} F_{jk}, \qquad \varepsilon \cdot F = \varepsilon_{ijkl} F^{ijkl} \tag{2}$$

If T_{ijk} vanishes, then F_{ijkl}, F_{ij} and F reduce respectively to the Riemann curvature tensor, Ricci tensor and Ricci scalar. In that case one has $F_{ij} = F_{ji}$, otherwise this tensor generically has both symmetric and antisymmetric parts. Also, F_{ijkl} is antisymmetric with respect to the interchange $i \leftrightarrow j$ and $k \leftrightarrow l$. Unlike the Riemann tensor of the standard torsion free connection, F_{ijkl} is not symmetric with respect to the interchange of the first pair of its indices with the last pair.

The tensor $T_{ijk} = -T_{ikj}$ can be decomposed into its irreducible components under the action of the local $O(1,3)$ transformations. They are given by

$$T_{ijk} = \frac{2}{3}(t_{ijk} - t_{ikj}) + \frac{1}{3}(\eta_{ij} v_k - \eta_{ik} v_j) + \varepsilon_{ijkl} a^l \tag{3}$$

where the field t_{ijk} is symmetric with respect to the interchange of i and j and satisfies the cyclic and trace identities,

$$t_{ijk} + t_{jki} + t_{kij} = 0, \qquad \eta^{ij} t_{ijk} = 0, \qquad \eta^{ik} t_{ijk} = 0$$

Due to the cyclic identity, all components of t_{ijk} can be expressed in terms of the antisymmetric part $t_{i[jk]}$,

$$t_{ijk} = \frac{2}{3}\left(t_{i[jk]} + t_{j[ik]}\right) \tag{4}$$

The antisymmetric tensor $t_{i[jk]}$ also obeys the cyclic and trace identities,

$$t_{i[jk]} + t_{j[ki]} + t_{k[ij]} = 0, \qquad \eta^{ij} t_{i[jk]} = 0\ .$$

Making use of the latter property, one finds that $t_{i[jk]}$, and hence t_{ijk}, has 16 independent components[3]. Together with 4 components of the vector v_i and 4 components of pseudo-vector a_i these make 24 independent components of $T_{ijk} = -T_{ikj}$, as they should.

Now we have all the algebraic preliminaries to define the general class of the actions of interest to us. The action is given by $S = \int d^4x\ e\ L$, where $e = \det e^i_\mu$ and $L = L_F + L_T$, with[4]

$$L_F = c_1 F + c_2 + c_3 F_{ij} F^{ij} + c_4 F_{ij} F^{ji} + c_5 F^2 + c_6 (\varepsilon_{ijkl} F^{ijkl})^2 + b F_{ijkl} F^{ijkl}, \tag{5}$$

[3]To see this, one notices that the cyclic identities for $t_{i[jk]}$ are trivially satisfied if any two of the indices are equal to each other. Hence, there are $4!/3! = 4$ non-trivial cyclic identities. Trace identities, instead, involve $t_{i[jk]}$ with two indices equal to each other, and there are 4 such identities. Hence, 8 out of 24 components of $t_{i[jk]}$ are not independent, which leaves 16 independent components.

[4]The mass terms in L_T can also be motivated in terms of spontaneous breaking of the gauge symmetry $SO(1,3)$ [24].

$$L_T = \alpha t_{ijk}t^{ijk} + \beta v_i v^i + \gamma a_i a^i \tag{6}$$

Note that an extra term of the form $F_{ijkl}F^{klij}$ can also be included in (5). However, using the 4-dimensional Gauss-Bonnet invariant, this term can be expressed through the other quadratic curvature invariants already included in (5). The defining property of the class of theories (5), (6) is that the derivative terms are geometric (they are expressed through the curvature tensor) while non-derivative terms are quadratic in torsion; the expression (5) is the most general quadratic expression (assuming parity conservation).

For the special case of $c_2 = 0$ and in Minkowski background, the spectrum of propagating modes has been studied in detail in Refs. [17, 18, 19, 20]. These studies have shown that in order to obtain a tachyon- and ghost-free perturbation spectrum about Minkowski space, the parameters of this action should be restricted in some specific manner. In this way all possible tachyon- and ghost-free theories have been tabulated. Among various possibilities there is a class of models which have, in addition to the massless graviton, a massive spin-2 propagating degree of freedom. This massive spin-2 mode originates from the t_{ijk}-component of the torsion tensor. In this note we consider a model belonging to the latter class. In Minkowski background this model has massless graviton, massive spin-2 mode as well as massive spin-0 propagating degree of freedom. This is achieved by setting $b = c_3 + c_4 + 3c_5 = 0$ and $\alpha = -\beta = \frac{4\gamma}{9}$. The parameter c_2 has been set equal to zero in Refs. [17, 18, 19, 20] but we keep it different from zero. This will enable us to have (anti-)de Sitter space as a solution to the field equations. Our Lagrangian $L = L_F + L_T$ is therefore given by

$$L_F = c_1 F + c_2 + c_3 F_{ij}F^{ij} + c_4 F_{ij}F^{ji} + c_5 F^2 + c_6(\varepsilon_{ijkl}F^{ijkl})^2,$$

$$L_T = \alpha\left(t_{ijk}t^{ijk} - v_i v^i + \frac{9}{4}a_i a^i\right) ,$$

where the only restriction is

$$c_3 + c_4 + 3c_5 = 0 \, .$$

We will show that the properties described above — the Einstein spaces as solutions to the field equations and the absence of both the Boulware–Deser phenomenon and superluminal propagation in maximally symmetric spaces — hold without any further restrictions on the parameters $c_1, \ldots, c_6$ and α entering the Lagrangian beyond what is required in flat background [5]. The masses of the spin-2 and spin-0 modes will pick up contributions due to the cosmological constant c_2 and will remain non-tachyonic if they are non-tachyonic

[5]For the class of models we consider in this paper the absence of ghosts and tachyons in the flat background require that $c_5 < 0, c_6 > 0, \alpha < 0$, and $\tilde{\alpha} = \alpha + \frac{2}{3}c_1 > 0$. Note that $c_1 = M_p^2$.

in Minkowski space. except for a stronger bound on c_5 in the de Sitter background. In order for the spin zero particle to be non tachyonic in de Sitter space c_5 will have to be bounded from above. This bound reduces to that of the flat space bound when the curvature vanishes. This is discussed in section 5.

The rest of this paper is organized as follows. In section 2 we present the full non-linear field equations in the model we study. In section 3 we show that torsion-free Einstein spaces are solutions to these equations. In section 4 we analyze the perturbations associated with fields v^i and a^i, about arbitrary Einstein backgrounds. In section 5 we study the perturbations originating from the field t_{ijk}, about maximally symmetric spaces. We conclude in section 6.

2 Field equations

In the first place, let us present the field equations in our model. We begin with the gravitational field equations.

2.1 Gravitational field equations

The gravitational field equations are found, as usual, from $\frac{\delta S}{\delta e^i_m} = 0$. This leads to

$$\begin{aligned} c_1 F_{ji} + c_3({F^m}_i F_{mj} - {F_j}^{mn}{}_i F_{mn}) + c_4({F^m}_i F_{jm} - {F_j}^{mn}{}_i F_{nm}) + 2c_5 F_{ji} F \\ + 2c_6 \varepsilon_{kmnj} {F^{kmn}}_i (\varepsilon_{rpqs} F^{rpqs}) + (D^k + v^k) F_{ijk} + H_{ij} - \frac{1}{2}\eta_{ij} L = 0 \end{aligned} \tag{7}$$

where

$$\begin{aligned} F_{ijk} &= \alpha\left[(t_{ijk} - t_{ikj}) - (\eta_{ij} v_k - \eta_{ik} v_j) - \frac{3}{4}\varepsilon_{ijkl} a^l\right] \\ H_{ij} &= T_{mni} {F^{mn}}_j - \frac{1}{2} T_{jmn} {F_i}^{mn} \end{aligned}$$

and D_i is the covariant derivative with respect to the connection A_{ijk}. Note that H_{ij} is second order in the torsion, therefore, in torsion-free backgrounds it will not contribute to the linearized equations for perturbations. Since we are going to consider only such backgrounds, this tensor will not play any role in our calculations. Nevertheless we keep it for the time being. We also note that the above equations have both symmetric and antisymmetric parts.

The antisymmetric part of the gravitational field equations (7) is straightforward to obtain. It reads

$$c_1 F_{[ji]} + \frac{c_4}{2}(F^m{}_i F_{jm} - F^m{}_j F_{im}) - \frac{1}{2}(F_j{}^{mn}{}_i - F_i{}^{mn}{}_j)(c_3 F_{mn} + c_4 F_{nm})$$
$$+2c_5 F_{[ji]} F + c_6(\varepsilon_{kmnj} F^{kmn}{}_i - \varepsilon_{kmni} F^{kmn}{}_j)(\varepsilon_{rpqs} F^{rpqs})$$
$$+(D^k + v^k) F_{[ij]k} + H_{[ij]} = 0 \ . \tag{8}$$

Taking the trace of eq. (7) gives a useful constraint on F, namely,

$$c_1 F = -3\alpha \nabla_i v^i - L_T - 2c_2$$

where ∇_i is the covariant derivative with respect to the spin connection $\omega^i_{j\mu}$ derived from the vierbein e^i_μ. Let us point out that for a torsion-free solution to the field equations, F is a constant given by

$$F = -\frac{2c_2}{c_1} \ . \tag{9}$$

For later use we also record the linear part of F in a torsion-free background,

$$c_1 F_{(1)} = -3\alpha \nabla_i v^i$$

We do not use special notation for the background objects, unless there is a risk of an ambiguity; the subscript (1) refers to linearized perturbations. For torsion-free backgrounds discussed in this paper, the torsion components v^i, a^i and t_{ijk} are perturbations by themselves; we do not label them by the subscript (1). Hereafter the operator ∇_i always denotes the covariant derivative with respect to the background metric.

Another useful formula is the first order relation between $F_{(1)}$ and the scalar curvature $R_{(1)}$ in a torsion-free background, where $R_{(1)}$ is derived from the spin connection $\omega^i_{j\mu}$. By making use of the definition of F and expanding in veirbein perturbations and torsion, one finds

$$F_{(1)} = R_{(1)} + 2\nabla_i v^i$$

Combining the last two equations we obtain the first order relation between $R_{(1)}$ and v in a torsion-free background, namely,

$$c_1 R_{(1)} = -3\tilde{\alpha} \nabla_i v^i$$

where $\tilde{\alpha}$ is a useful parameter which will be encountered frequently in the sequel,

$$3\tilde{\alpha} = 2c_1 + 3\alpha$$

2.2 Torsion field equations

We now move on to the torsion field equations. These are obtained from $\frac{\delta S}{\delta A^i_{j\mu}} = 0$. Written in the orthonormal basis they become

$$
\begin{aligned}
& c_3 \left\{ \eta^{ik}(D_m + \frac{2}{3}v_m)F^{jm} - \eta^{jk}(D_m + \frac{2}{3}v_m)F^{im} - (D^i + \frac{2}{3}v^i)F^{jk} + (D^j + \frac{2}{3}v^j)F^{ik} \right\} \\
& + c_4 \left\{ \eta^{ik}(D_m + \frac{2}{3}v_m)F^{mj} - \eta^{jk}(D_m + \frac{2}{3}v_m)F^{mi} - (D^i + \frac{2}{3}v^i)F^{kj} + (D^j + \frac{2}{3}v^j)F^{ki} \right\} \\
& + c_5 \left\{ \eta^{ik}(D^j + \frac{2}{3}v^j)F - \eta^{jk}(D^i + \frac{2}{3}v^i)F \right\} + 4c_6 \left\{ \varepsilon^{ijkm}(D_m + \frac{2}{3}v_m)(\varepsilon \cdot F) \right\} \\
& - \left(\frac{4}{3}t^k{}_{[mn]} + \varepsilon^k{}_{mnp}a^p \right) \left\{ c_3(\eta^{im}F^{jn} - \eta^{jm}F^{in}) + c_4(\eta^{im}F^{nj} - \eta^{mj}F^{ni}) \right. \\
& \left. + 2c_5\eta^{im}\eta^{jn}F + 2c_6\varepsilon^{ijkm}(\varepsilon \cdot F) \right\} + H^{ijk} = 0 \qquad (10)
\end{aligned}
$$

where

$$H_{ijk} = -\tilde{\alpha}(t_{kij} - t_{kji}) + \tilde{\alpha}(\eta_{ki}v_j - \eta_{kj}v_i) - \frac{3\tilde{\alpha}}{2}\varepsilon_{ijkl}a^l$$

Equations (10) can be decomposed exactly in the same manner as the torsion itself has been decomposed into t_{ijk}, v_i and a_i. This is done by taking the trace of eq. (10) over j and k and by contracting eq. (10) with ε_{ijkl}. In the course of these manipulations we make use of the constraint $c_3 + c_4 = -3c_5$.

The trace of the torsion equation is obtained by contracting eq. (10) with η^{jk},

$$
\begin{aligned}
& -3c_5 \left(D_j F^{(ij)} - \frac{1}{2}D^i F \right) - 2c_5 \left(V_j F^{(ij)} - \frac{1}{2}V^i F \right) + (c_3 - c_4)D_j F^{[ij]} \\
& + \frac{2}{3}(c_3 - c_4)V_i F^{[ij]} - 3c_5 t^{i(jn)}F_{(jn)} + \frac{1}{3}(c_3 - c_4)t^{i[jn]}F_{[jn]} \\
& - \frac{1}{2}(c_3 - c_4)\varepsilon^{ijnl}a_l F_{[jn]} + 6c_6 a^i(\varepsilon \cdot F) + \frac{3}{2}\tilde{\alpha}v^i = 0 \qquad (11)
\end{aligned}
$$

The curl of the torsion equation is found by contracting eq. (10) with ε_{ijkl},

$$
\begin{aligned}
& (c_3 - c_4)\varepsilon_{lijk}D^i F^{jk} - 12c_6 D_l(\varepsilon \cdot F) - \frac{2}{3}\varepsilon_{ijkl}t_n{}^{ik}(c_3 F^{jn} + c_4 F^{nj}) - 8c_6 v_l(\varepsilon \cdot F) \\
& - \frac{2}{3}(c_3 - c_4)\varepsilon_{ijkl}v^i F^{jk} - 2(c_3 F_{jl} + c_4 F_{lj})a^j + \frac{2}{9}\tilde{\alpha}a_l = 0 \qquad (12)
\end{aligned}
$$

3 Backgrounds

As mentioned above, we consider torsion-free backgrounds only. In this case $F_{ijkl} = R_{ijkl}, F_{ij} = R_{ij}$ and $F = R$, where R_{ijkl}, R_{ij} and R denote, respectively, the Riemann tensor, the Ricci tensor and the Ricci scalar of the background metric.

The gravitational equations (7) reduce to

$$c_1 R_{ij} - 3c_5 (R^m{}_i R_{mj} - R_j{}^{mn}{}_i R_{mn}) - 4\frac{\lambda c_5}{c_1} R_{ij} - \frac{1}{2}\eta_{ij} L = 0 \tag{13}$$

where

$$L = -3c_5 R_{ij} R^{ij} + 4c_5 \left(\frac{c_2}{c_1}\right)^2 - c_2$$

These equations should be solved alongside with the torsion equation (10) which yields

$$\nabla_i R_{jk} - \nabla_j R_{ik} = 0 \tag{14}$$

In writing these equations we made use of the fact that scalar curvature of the background manifold is constant, see (9). Combining eqs. (13) and (14) with the Bianchi identity we obtain a condition on the Riemann tensor or equivalently on the Weyl tensor of the background manifold, namely,

$$\nabla^i R_{ijkl} = 0 = \nabla^i W_{ijkl}$$

The Weyl tensor is defined by

$$R_{ijkl} = W_{ijkl} + \frac{1}{2}(\eta_{ik} R_{jl} - \eta_{il} R_{jk} - \eta_{jk} R_{il} + \eta_{jl} R_{ik}) - \frac{1}{6}(\eta_{ik}\eta_{jl} - \eta_{il}\eta_{jk}) R$$

The tensor W_{ijkl} has all the symmetries of the Riemann tensor plus the additional property that it is trace-free in all pairs of indices.

By inspecting eqs. (13) and (14) we find that they are identically satisfied for Einstein manifolds (whose definition is $R_{ij} = \mathrm{const} \cdot \eta_{ij}$). For such manifolds we obtain

$$R_{ijkl} = \Lambda(\eta_{ik}\eta_{jl} - \eta_{il}\eta_{jk}) + W_{ijkl}, \qquad R_{ij} = 3\Lambda\eta_{ij}, \qquad R = 12\Lambda,$$

and the only constraint we arrive at is

$$\Lambda = -\frac{c_2}{6c_1} \ . \tag{15}$$

In this respect our theory, unlike some other theories whose Lagrangians are of the second order in the Riemann tensor, is similar to General Relativity.

4 Linearized theory in Einstein backgrounds

In this section we study field perturbations about general torsion-free Einsten backgrounds. We will be able to go pretty far in our analysis. Namely, we will show that the vector field

v_i does not have its own propagating modes, as it is expressed through the field t_{ijk}. We will also show that the field a_i is a gradient, and its longitudinal part obeys the massive Klein–Gordon equation. These properties hold for arbitrary Einstein space background and are exactly the same as in the theory about Minkowski background. On the other hand, to find explicitly the spectrum of perturbations associated with the field t_{ijk}, we will have to resort to maximally symmetric backgrounds. This is done in section 5.

4.1 Consequences of gravitational equations

4.1.1 Antisymmetric gravitational equations

Important constraints are obtained by considering antisymmetric part of the gravitational equations. It is straightforward to show that in the backgrounds of Einstein manifolds eq. (8) reduces to

$$(c_1 - 4\Lambda c_3)F_{(1)[ji]} - \nabla^k F_{(1)[ji]k} - (c_3 - c_4)W_j{}^{[mn]}{}_i F_{(1)[mn]} = 0 \ . \tag{16}$$

By linearizing F_{ij} defined in (2) we find

$$\begin{aligned} F_{(1)[ji]} &= -\frac{2}{3\alpha}\nabla^k F_{[ji]k} \\ &= \frac{2}{3}(\nabla^k t_{k[ji]} - \nabla_{[j}v_{i]} + \frac{3}{4}\varepsilon_{jikl}\nabla^k a^l) \end{aligned} \tag{17}$$

Hence, eq. (16) takes the following form,

$$(I_i{}^{[mn]}{}_j + aW_i{}^{[mn]}{}_j)(\nabla^k t_{k[mn]} - \nabla_{[m}v_{n]}) = -\frac{3}{4}(\varepsilon_{ijkl} + aW_i{}^{[mn]}{}_j\varepsilon_{mnkl})\nabla^k a^l \tag{18}$$

where

$$\begin{aligned} a &= -\frac{\frac{2}{3}(c_3 - c_4)}{\alpha + \frac{2}{3}(c_1 - 4\Lambda c_3)} \\ I_i{}^{[mn]}{}_j &= \frac{1}{2}(\delta_i^m\delta_j^n - \delta_i^n\delta_j^m) \end{aligned} \tag{19}$$

Equation (18) is solved by

$$\nabla^k t_{k[ij]} - \nabla_{[i}v_{j]} = -\frac{3}{4}\varepsilon_{ijkl}\nabla^k a^l \tag{20}$$

Inserting this in eq. (17) we obtain

$$\nabla^k F_{[ij]k} = 0$$

and hence

$$F_{(1)[ij]} = 0 \tag{21}$$

This result plays a crucial role in simplifying all other equations.

4.1.2 Symmetric gravitational equations

Inserting the results of section 4.1.1 in the gravitational field equations (7) and linearizing in the Einstein background we obtain the equations for the symmetric components[6] of $F_{(1)(ij)} = F_{(1)ij}$. These equations read

$$c_1 \left(F_{(1)ij} - \frac{1}{2}\eta_{ij} F_{(1)} \right) + \nabla^k F_{(1)ijk} + 3c_5 W_{jmni} F^{mn}_{(1)} = 0 \tag{22}$$

Linearizing F_{ij} and recalling the definition of F_{ijk} we write

$$\begin{aligned} F_{(1)ij} &= R_{(1)ij} - 2\nabla^k t_{k(ij)} + \frac{1}{3}(\nabla_i v_j + \nabla_j v_i) + \frac{1}{3}\eta_{ij} \nabla \cdot v && (23) \\ \nabla^k F_{(1)ijk} &= -\alpha \left[3\nabla^k t_{k(ij)} - \frac{1}{2}(\nabla_i v_j + \nabla_j v_i) + \eta_{ij} \nabla \cdot v \right] && (24) \end{aligned}$$

Substitution in (22) yields

$$c_1 R_{(1)ij} = -\frac{\tilde{\alpha}}{2}\eta_{ij} \nabla \cdot v + 3\tilde{\alpha}\nabla^k t_{k(ij)} - \frac{\tilde{\alpha}}{2}(\nabla_i v_j + \nabla_j v_i) - 3c_5 W_{imnj} F^{mn}_{(1)} \tag{25}$$

This equation can be viewed as the relationship between the gravitational perturbation contained in $R_{(1)ij}$ and perturbations of the torsion field. Therefore, eq. (25) is irrelevant for the study of the spectrum of the torsion perturbations. This spectrum comes out from the analysis of the torsion field equations. Of course, the homogeneous part of eq. (25) describes the propagation of massless gravitons in the Einstein backgrounds, just like in General Relativity.

4.2 Linearized torsion field equations

4.2.1 The pseudovector a_l

Substituting (21) in eq. (12) and using the fact that $\varepsilon \cdot F = 6\nabla^i a_i$ we obtain the following linearized equation for a_l,

$$8c_6 \nabla_l (\nabla \cdot a) - \left(2\Lambda c_5 + \frac{\tilde{\alpha}}{2} \right) a_l = 0 \tag{26}$$

This equation shows that the pseudovector field a_l is a gradient. As a result the right hand side of (20) vanishes and eq. (20) becomes the important constraint

$$\nabla^k t_{k[mn]} = \nabla_{[m} v_{n]} \tag{27}$$

[6] Note that, whenever convenient, we drop the symmetrization bracket (ij) from $F_{(1)(ij)}$ and $\nabla^k F_{(1)(ij)k}$ as the antisymmetric parts of these tensors are zero to the first order.

Furthermore by acting with ∇^l on (26) we obtain the Klein-Gordon equation for the longitudinal part $\sigma = \nabla^i a_i$ of the pseudovector field,

$$\left(\nabla^2 - \frac{2\Lambda c_5 + \frac{\tilde{\alpha}}{2}}{8c_6}\right)\sigma = 0 \tag{28}$$

It can be verified that the mass of the spin zero field σ coincides with the flat space result when $\Lambda = 0$. Thus, the only propagating degree of freedom of the field a_i is its longitudinal part σ. Its transverse part is simply zero.

4.2.2 The vector field v_i

Next we obtain the equation for the vector field v_i. To this end we substitute $F_{[ij]} = 0$ in the linearized version of eq. (11). The result is

$$\left(D_j F^{(ij)} - \frac{1}{2}D^i F\right)_{(1)} = \left(2\Lambda + \frac{\tilde{\alpha}}{2c_5}\right)v^i \tag{29}$$

One can show that

$$(D^j F_{(ij)})_{(1)} = \nabla^i F_{(1)(ij)}$$

If the torsion field were absent, the left hand side of (29) would be the linearized covariant derivative of the Einstein tensor and would vanish by virtue of the contracted Bianchi identity. In the presence of torsion the Bianchi identity is modified and is given by

$$D_k F_{ijlm} + T^n{}_{kl} F_{ilmn} + \text{cyclic }(klm) = 0 \tag{30}$$

Contracting this identity we obtain

$$D^i F_{ij} - \frac{1}{2}D_j F = T^i{}_{kj} F^k{}_i + \frac{1}{2}T^i{}_{kl} F^{kl}{}_{ji} \tag{31}$$

We note that the identitites (30) and (31) are valid in the full theory. Linearizing the identity (31) in the Einstein space background, we obtain

$$D^i F_{ij} - \frac{1}{2}D_j F = 2\Lambda v_j + \frac{1}{2}T^i{}_{kl} W^{kl}{}_{ji} \tag{32}$$

The comparison of (29) and (32) gives

$$v_i = \frac{c_5}{\tilde{\alpha}} T^j{}_{kl} W^{kl}{}_{ij}$$

Upon substituting for T_{ijk} from (3) we finally obtain

$$v_i = \frac{4c_5}{3\tilde{\alpha}} W_{ijkl} t^{j[kl]} \tag{33}$$

Equation (33) shows that the vector field v_i is not an independently propagating field, exactly as in the flat space. In the general Einstein space backgrounds the field v_i is determined in terms of the tensor field t_{ijk}.

It is useful to note that we can obtain a constraint on the divergence of the field $t_{i(jk)}$ by combining (27) and (33) with the commutators of covariant derivatives. We give this relation here for later reference,

$$6\nabla^i\nabla^k t_{k(ij)} = \nabla^2 v_j - \nabla_j \nabla \cdot v - \left(3\Lambda + \frac{3\tilde{\alpha}}{2c_5}\right) v_j \tag{34}$$

4.2.3 Equation for t_{ijk}

Using the symmetries of t_{ijk} one can show that

$$[\nabla_k, \nabla_j]t^{jik} = W^{jkil} t_{lkj}$$

This relation is instrumental in proving that

$$\nabla^j\nabla^k t_{ikj} = -\frac{1}{2}\nabla^j\nabla^k t_{kji} - \frac{1}{2}W_{jkli}t^{klj}$$

We now make use of the results of sections 4.1, 4.2.1, 4.2.2 and rewrite the torsion field equation (10) as the equation for the only remaining component, namely, for the field t_{ijk},

$$\begin{aligned} &\nabla_i F_{(1)jk} - \nabla_j F_{(1)ik} + \frac{1}{6}(\eta_{ik}\nabla_j F_{(1)} - \eta_{jk}\nabla_i F_{(1)}) \\ &-\frac{1}{3}\left(2\Lambda + \frac{\tilde{\alpha}}{2c_5}\right)\{(\eta_{ik}v_j - \eta_{jk}v_i) + 4t_{k[ij]}\} = 0 \end{aligned} \tag{35}$$

Here v_i should be expressed in terms of t_{ijk} according to (33). It can be verified that the trace of this equation over j and k as well as its divergence over k are zero. On the other hand if we apply ∇^i to it we obtain a second order equation which reads

$$\begin{aligned} &(\nabla^2 - 4\Lambda)F_{(1)jk} - W_{ijkl}F^{il}_{(1)} - \frac{1}{3}\left[\nabla_j\nabla_k + \frac{1}{2}\eta_{jk}(\nabla^2 - 6\Lambda)\right]F_{(1)} \\ &-\frac{1}{3}\left(2\Lambda + \frac{\tilde{\alpha}}{2c_5}\right)\{2(\nabla_k v_j + \nabla_j v_k) - \eta_{jk}\nabla\cdot v + 6\nabla^i t_{i(jk)}\} = 0 \end{aligned} \tag{36}$$

This equation is symmetric with respect to the interchange of j and k. It can be verified that it is also traceless as well as divergence free.

5 Symmetric spaces

In view of complexity of eq. (35), let us specify now to maximally symmetric backgrounds. For these spaces, the Weyl tensor vanishes, $W_{ijkl} = 0$. Then eq. (33) implies that

$$v_i = 0 \tag{37}$$

Substitution of this in (27) gives

$$\nabla^k t_{k[mn]} = 0$$

Using (37) in (23) we obtain

$$F_{(1)ij} = R_{(1)ij} - 2\nabla^k t_{k(ij)} \tag{38}$$

The linearized Einstein equations (25) then reduce to

$$c_1 R_{(1)ij} = 3\tilde{\alpha}\nabla^k t_{k(ij)} \tag{39}$$

Let us define the symmetric tensor field χ_{ij} by

$$\chi_{ij} = \nabla^k t_{k(ij)} \tag{40}$$

It is immediately seen that χ is traceless and transverse[7], i.e.,

$$\eta^{ij}\chi_{ij} = 0, \qquad \nabla^i \chi_{ij} = 0$$

Thus χ_{ij} has only five independent components. We now write R_{ij} in (39) in terms of χ_{ij} and substitute it in (38) so that $F_{(1)ij}$ is expressed in terms of χ_{ij},

$$F_{(1)ij} = \frac{3\alpha}{c_1}\chi_{ij} \tag{41}$$

We insert this expression in eq. (36) and obtain the following Klein-Gordon type equation for the field χ_{ij},

$$(\nabla^2 - M_2^2)\chi_{ij} = 0, \tag{42}$$

where M_2^2 is given by

$$M_2^2 = 4\Lambda\left(1 + \frac{c_1}{3\alpha}\right) + \frac{\tilde{\alpha}c_1}{3\alpha c_5} \tag{43}$$

Finally, the field $t_{k[ij]}$ is determined by eq. (35), which by making use of (41) is reduced to

$$t_{k[ij]} = \frac{9\alpha}{4c_1(2\Lambda + \frac{\tilde{\alpha}}{2})}(\nabla_i \chi_{jk} - \nabla_j \chi_{ik}) \tag{44}$$

[7]The transversality of χ_{ij} follows from (34) when we set $v_i = 0$ in that equation.

As we pointed out in section 1, the tensor t_{ijk} may be expressed through its antisymmetric part $t_{i[jk]}$, so the field χ_{ij} completely determines t_{ijk}.

According to its definition (40), the field χ_{ij} is a gauge-covariant, tensor field. It follows from eqs. (39), and (44) that both torsion field and metric perturbations (in particular, $R_{(1)ij}$) do not vanish for this mode. In other words, the massive tensor field results from mixing between torsion and metric. Due to this mixing, the massive tensor mode is sourced both by "spin" (source for torsion) and energy-momentum (source for metric). This is considered in some detail in Ref. [25]. Hence, the theory we discuss in this paper is a candidate for infrared modified gravity.

Since the massive tensor mode χ_{ij} is free of pathologies in symmetric backgrounds, its quadratic action in Miknowski background necessarily has the Fierz–Pauli structure. The novelty here is that the Fierz–Pauli equation is effectively deformed into curved backgrounds, and no pathologies (say, of the Boulware–Deser type) are introduced at least in the case of maximally symmetric backgrounds. The explicit form of the generalized Fierz–Pauli equation emerging in this theory is given in Ref. [25] for arbitrary Einstein backgrounds.

To sum up, in the maximally symmetric background, the only propagating modes are massless graviton, massive spin zero particle with a mass which can be read off from eq. (28) and massive spin-2 field χ_{ij}, the same as in the flat space. All masses reduce to the flat space values when $\Lambda = 0$.

The flat space analysis indicates that for the absence of ghosts and tachyons the parameters should satisfy certain inequalities which in our notations read,

$$c_5 < 0 \qquad c_6 > 0 \qquad \alpha < 0 \qquad \tilde{\alpha} > 0 \tag{45}$$

Note that $c_1 = M_p^2$ has to be positive. The parameter c_2 is the cosmological term in the action and does not enter the flat space calculations. We see from (15) that in order for Λ to be positive c_2 has to be negative. Thus with positive Λ, i.e. in de Sitter background, the spin zero field σ in eq. (28) will be non-tachyonic, if $4\Lambda c_5 + \tilde{\alpha} > 0$.

It is also known that there is a unitarity bound on the mass of the spin-2 field in de Sitter background [26]. Comparing our notation with that of Ref. [26] (especially our eq. (42) with eq. (3.10) of Ref. [26]) we conclude that in our terms the untarity bound is $M_2^2 > 4\Lambda$. Comparing the mass of the σ field with that of the spin 2 field we find that

$$M_2^2 = 4\Lambda + \frac{16c_6}{3\alpha c_5} M_0^2 c_1 \tag{46}$$

where M_0 denotes the mass of the σ field as defined by (28). Clearly our spin-2 mass will satisfy the unitarity bound as long as the σ field is non-tachyonic.

6 Conclusions

In this paper we have examined the consistency of the model with a massive spin-2 particle, whose Lagrangian involves quadratic terms in curvature and torsion tensors. This problem was analyzed a long time ago in flat space background. We have extended this analysis to curved background spaces. First we have given the linearized equations for the perturbations about arbitrary Einstein manifold and found that several components of the torsion field do not propagate, while metric perturbations correspond to massless propagating graviton. To analyze the remaining components of the torsion field, we then specified to maximally symmetric backgrounds. We have shown that, unlike in the Fierz-Pauli theory, in our model the number and the nature of the propagating modes do not change when the background becomes curved. The full analysis of the propagation in the background of an arbitrary Einstein manifold still needs to be carried out.

In maximally symmetric backgrounds, the propagating modes in our model obey the usual Klein–Gordon type equations. Hence, there is no superluminal propagation, again in contrast to the Fierz–Pauli theory.

There are several classes of such models which are tachyon- and ghost-free in Minkowski background in specific regions of their parameter space. It will be interesting to see if there are any other consistent subclasses among them.

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